Markowitz Efficient Portfolio Frontier as Least-Norm Analytic Solution to Underdetermined Equations

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Introduction

Modern portfolio theory deals in part with the efficient allocation of investments among risky assets to achieve high-return low-risk portfolios. Assets are often depicted on the expected-return–risk plane, where the axes are the expected return of a portfolio $E(r_P)$ versus the portfolio standard deviation of returns σ_P. A portfolio is represented by a weighting of risky assets (usually stocks) corresponding to the proportion of an investor's money in each asset.

One aspect of modern portfolio theory concerns the determination of the *efficient frontier*. This is the set of portfolios (i.e., weightings of all possible securities) that mark the lowest-deviation (left) boundary on the set of all feasible portfolios. Formally, the efficient frontier is the set of all *Markowitz efficient portfolios* wherein no change in the weightings of a security can increase the portfolio's expected return without increasing its variance. A closed-form expression for the efficient frontier can be determined through an application of the least-norm solution to underdetermined equations.

Problem Statement

We define the weighting vector of a portfolio $w \in \mathbb{R}^n$ as the vector whose elements represent the proportion of the investor's money in each of the market's n securities. Additionally, each pair of securities i and j has a covariance σ_{ij}. Simple statistics indicates that the total variance on returns of a portfolio P is given by

$$\sigma_P^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij}$$

In terms of the covariance matrix Σ, this is

$$\sigma_P^2 = w^T \Sigma w$$

We now define the vector of individual asset expected returns as $r \in \mathbb{R}^n$, where r_i is the expected return of the ith asset. Linearity of expectation dictates that the expected return r_P on a portfolio w is

$$r_P = r^T w$$

The formal statement of the problem follows:

Problem. For all feasible expected portfolio returns r_P, determine the lowest-variance portfolio, or the solution to

$$\min_w \sigma_P^2 = w^T \Sigma w$$

subject to

$$\begin{aligned} \sum_{i=1}^{n} w_i &= 1 \\ r^T w &= r_P \end{aligned}$$

We allow for negative weights, which correspond to short sales.

Solution with Least-Norm

The problem can be reformulated as follows. Consider the square-root of the covariance matrix $\Sigma^{1/2}$ satisfying $\Sigma^{1/2}\Sigma^{1/2} = \Sigma$. By the symmetry of Σ, we know that $\Sigma^{1/2}$ is also symmetric such that ${\Sigma^{1/2}}^T\Sigma^{1/2} = \Sigma$. We can now rewrite the objective function:

$$w^T\Sigma w = w^T{\Sigma^{1/2}}^T\Sigma^{1/2}w = (\Sigma^{1/2}w)^T\Sigma^{1/2}w = ||\Sigma^{1/2}w||^2$$

Let us also define a matrix $B \in \mathbb{R}^{2\times n}$ and vector $y \in \mathbb{R}^2$ as follows:

$$B = \begin{bmatrix} r^T \\ e^T \end{bmatrix} \qquad y = \begin{bmatrix} r_P \\ 1 \end{bmatrix}$$

We can rewrite the constraints as

$$Bw = y$$

Now, we define the inverse of $\Sigma^{1/2}$ as $\Sigma^{-1/2}$. Rewriting the above:

$$Bw = B\Sigma^{-1/2}(\Sigma^{1/2}w) = y$$

Now, we define $x = \Sigma^{1/2}w$ and $A = B\Sigma^{-1/2}$ and reformulate the problem:

Problem.

$$\min_x ||x||^2$$

subject to

$$Ax = y$$

This is a least-norm problem. The solution is

$$\begin{aligned} x^* &= A^T(AA^T)^{-1}y \\ &= (B\Sigma^{-1/2})^T(B\Sigma^{-1/2}(B\Sigma^{-1/2})^T)^{-1}y \\ &= \Sigma^{-1/2}B^T(B\Sigma^{-1/2}\Sigma^{-1/2}B^T)^{-1}y \end{aligned}$$

The optimal weights are then given by

$$w^* = \Sigma^{-1/2}x^* = \Sigma^{-1/2}\Sigma^{-1/2}B^T(B\Sigma^{-1/2}\Sigma^{-1/2}B^T)^{-1}y$$

We note, however, that $\Sigma^{-1/2}\Sigma^{-1/2} = \Sigma^{-1}$ is a left and right inverse of Σ. With this, we can rewrite the expression for the optimal weights in terms of B and Σ^{-1}:

$$w^* = \Sigma^{-1}B^T(B\Sigma^{-1}B^T)^{-1}y$$

In general, the optimal weights are a function of the desired expected return r_P. The optimal weights are then

$$w^*(r_P) = \Sigma^{-1}B^T(B\Sigma^{-1}B^T)^{-1}\begin{bmatrix} r_P \\ 1 \end{bmatrix} \tag{1}$$

This result yields a remarkable fact: Any portfolio on the efficient frontier is a linear combination of two market portfolios whose weights are the columns of the matrix $M = \Sigma^{-1}B^T(B\Sigma^{-1}B^T)^{-1}$. Specifically, we have that

$$w^*(r_P) = r_P m_1 + m_2$$

where

$$\begin{bmatrix} m_1 & m_2 \end{bmatrix} = \Sigma^{-1}B^T(B\Sigma^{-1}B^T)^{-1}$$

Furthermore, the market portfolio represented by m_1 is a zero-position portfolio; i.e., its weights add to zero. The weights of the market portfolio represented by m_2 sum to unity. To see this, consider the (undesirable) portfolio on the frontier for which $r_P = 0$. Clearly, this portfolio consists solely of m_2, and since the weights of this portfolio must sum to unity, we have that the weights m_2 sum to unity. For any desired return r_P, the optimal portfolio weights are $w^*(r_P) = r_P m_1 + m_2$. Since both the weights $w^*(r_P)$ and m_2 sum to unity for all r_P, we have that the weights m_1 sum to zero.

The variance of the portfolio admits an elegant expression:

$$\begin{aligned} \sigma_P^2(r_P) &= w^*(r_P)^T \Sigma w^*(r_P) \\ &= \left[\Sigma^{-1}B^T(B\Sigma^{-1}B^T)^{-1}y\right]^T \Sigma\Sigma^{-1}B^T(B\Sigma^{-1}B^T)^{-1}y \\ &= y^T(B\Sigma^{-1}B^T)^{-1}B\Sigma^{-1}\Sigma\Sigma^{-1}B^T(B\Sigma^{-1}B^T)^{-1}y \\ &= y^T(B\Sigma^{-1}B^T)^{-1}B\Sigma^{-1}B^T(B\Sigma^{-1}B^T)^{-1}y \\ &= y^T(B\Sigma^{-1}B^T)^{-1}y \end{aligned}$$

Finally, the equation for the efficient frontier in the expected-return–risk plane is

$$\sigma_P(r_P) = \sqrt{\begin{bmatrix} r_P \\ 1 \end{bmatrix}^T (B\Sigma^{-1}B^T)^{-1} \begin{bmatrix} r_P \\ 1 \end{bmatrix}} \tag{2}$$

The variance $\sigma_P^2(r_P)$ of a Markowitz efficient portfolio is a quadratic function of r_P given by

$$\sigma_P^2(r_P) = ar_P^2 + 2br_P + c$$

where

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} = (B\Sigma^{-1}B^T)^{-1}$$

Thus, we need only compute $a, b, c \in \mathbb{R}$ and $M \in \mathbb{R}^{n\times 2}$ once for the economy, and the entire efficient frontier along with the composition of its portfolios can be obtained readily with only marginal computation.

Example

Let us consider a simple example. Envision a rudimentary stock market consisting of twenty securities. Suppose we have estimated the covariance matrix using the capital asset pricing model. We then apply Equation 2 to find the efficient frontier, shown in Figure 1. Points lying to the right of this curve represent inefficient portfolios, while points lying to the left represent infeasible portfolios. The dotted section of the curve lying below the turning point of minimum variance is also inefficient, since for any portfolio along this section, there exists a portfolio with equal variance but higher return.

The covariance matrix of the securities in this imaginary economy is given by

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}^T & \Sigma_{22} \end{bmatrix}$$

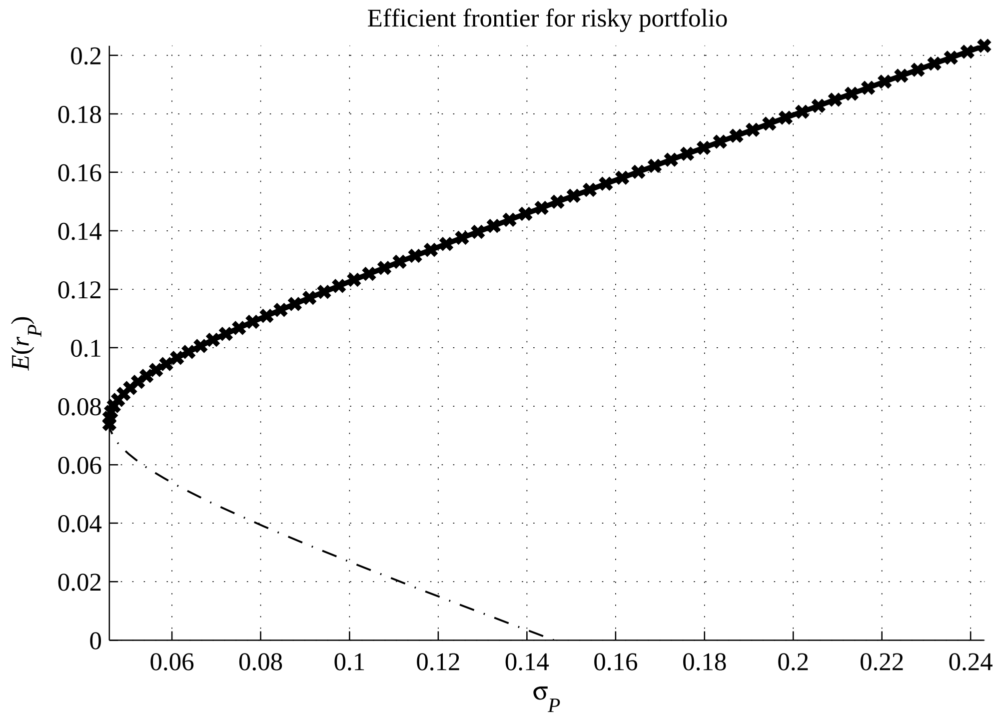

Figure 1: Efficient frontier for rudimentary stock market

where

$$\Sigma_{11} = \begin{bmatrix} 0.3055 & 0.0024 & 0.0614 & -0.0026 & -0.0877 & 0.0030 & -0.0011 & -0.0156 & 0.0010 & -0.0983 \\ 0.0024 & 0.3355 & 0.0624 & -0.0018 & -0.0897 & 0.0037 & -0.0006 & -0.0175 & 0.0006 & -0.1015 \\ 0.0614 & 0.0624 & 3.7843 & -0.0277 & 0.0753 & -0.0232 & -0.0179 & 0.0687 & 0.0124 & 0.1178 \\ -0.0026 & -0.0018 & -0.0277 & 0.2740 & 0.0394 & -0.0011 & 0.0007 & 0.0064 & -0.0006 & 0.0438 \\ -0.0877 & -0.0897 & 0.0753 & 0.0394 & 2.4313 & 0.0344 & 0.0257 & -0.1036 & -0.0177 & -0.1965 \\ 0.0030 & 0.0037 & -0.0232 & -0.0011 & 0.0344 & 0.5340 & -0.0009 & 0.0101 & 0.0006 & 0.0410 \\ -0.0011 & -0.0006 & -0.0179 & 0.0007 & 0.0257 & -0.0009 & 0.3091 & 0.0047 & -0.0003 & 0.0288 \\ -0.0156 & -0.0175 & 0.0687 & 0.0064 & -0.1036 & 0.0101 & 0.0047 & 0.6257 & -0.0030 & -0.1266 \\ 0.0010 & 0.0006 & 0.0124 & -0.0006 & -0.0177 & 0.0006 & -0.0003 & -0.0030 & 0.2907 & -0.0198 \\ -0.0983 & -0.1015 & 0.1178 & 0.0438 & -0.1965 & 0.0410 & 0.0288 & -0.1266 & -0.0198 & 1.4350 \end{bmatrix}$$

$$\Sigma_{12} = \begin{bmatrix} -0.0046 & 0.0592 & -0.0322 & 0.0120 & -0.0210 & -0.0061 & 0.0001 & -0.0493 & -0.0009 & -0.0690 \\ -0.0035 & 0.0604 & -0.0347 & 0.0129 & -0.0193 & -0.0052 & -0.0001 & -0.0502 & -0.0012 & -0.0600 \\ -0.0348 & -0.0449 & 0.0882 & -0.0332 & -0.0577 & -0.0323 & 0.0071 & 0.0334 & 0.0084 & -0.3045 \\ 0.0025 & -0.0267 & 0.0138 & -0.0051 & 0.0103 & 0.0032 & -0.0001 & 0.0223 & 0.0003 & 0.0351 \\ 0.0492 & 0.0808 & -0.1362 & 0.0513 & 0.0780 & 0.0452 & -0.0103 & -0.0614 & -0.0124 & 0.4224 \\ -0.0010 & -0.0228 & 0.0170 & -0.0063 & 0.0028 & -0.0003 & 0.0005 & 0.0187 & 0.0009 & 0.0009 \\ 0.0013 & -0.0173 & 0.0095 & -0.0035 & 0.0060 & 0.0017 & -0.0000 & 0.0144 & 0.0003 & 0.0195 \\ 0.0068 & 0.0681 & -0.0571 & 0.0214 & -0.0009 & 0.0046 & -0.0021 & -0.0554 & -0.0035 & 0.0332 \\ -0.0010 & 0.0120 & -0.0064 & 0.0024 & -0.0044 & -0.0013 & 0.0000 & -0.0100 & -0.0002 & -0.0146 \\ 0.0540 & 0.1229 & -0.1725 & 0.0648 & 0.0785 & 0.0486 & -0.0117 & -0.0957 & -0.0147 & 0.4482 \end{bmatrix}$$

$$\Sigma_{22} = \begin{bmatrix} 0.2368 & -0.0334 & 0.0160 & -0.0059 & 0.0144 & 0.0047 & -0.0003 & 0.0280 & 0.0003 & 0.0512 \\ -0.0334 & 3.0223 & 0.0885 & -0.0333 & -0.0542 & -0.0309 & 0.0069 & 0.0367 & 0.0083 & -0.2897 \\ 0.0160 & 0.0885 & 0.7725 & 0.0322 & 0.0125 & 0.0128 & -0.0041 & -0.0713 & -0.0060 & 0.1093 \\ -0.0059 & -0.0333 & 0.0322 & 0.8388 & -0.0045 & -0.0047 & 0.0015 & 0.0269 & 0.0022 & -0.0403 \\ 0.0144 & -0.0542 & 0.0125 & -0.0045 & 0.0405 & 0.0155 & -0.0020 & 0.0463 & -0.0012 & 0.1583 \\ 0.0047 & -0.0309 & 0.0128 & -0.0047 & 0.0155 & 0.1766 & -0.0005 & 0.0260 & 0.0000 & 0.0579 \\ -0.0003 & 0.0069 & -0.0041 & 0.0015 & -0.0020 & -0.0005 & 0.3567 & -0.0057 & -0.0001 & -0.0061 \\ 0.0280 & 0.0367 & -0.0713 & 0.0269 & 0.0463 & 0.0260 & -0.0057 & 3.6755 & -0.0068 & 0.2443 \\ 0.0003 & 0.0083 & -0.0060 & 0.0022 & -0.0012 & 0.0000 & -0.0001 & -0.0068 & 0.5097 & -0.0013 \\ 0.0512 & -0.2897 & 0.1093 & -0.0403 & 0.1583 & 0.0579 & -0.0061 & 0.2443 & -0.0013 & 0.7406 \end{bmatrix}$$

The vector of returns is

$$r = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$$

where

$$\begin{aligned} r_1 &= \begin{bmatrix} 0.1037 & 0.0940 & 0.0571 & 0.0650 & 0.1156 & 0.1807 & 0.1893 & 0.0536 & 0.1663 & 0.1857 \end{bmatrix}^T \\ r_2 &= \begin{bmatrix} 0.1666 & 0.2028 & 0.1550 & 0.1688 & 0.1052 & 0.1914 & 0.1056 & 0.0595 & 0.1649 & 0.2033 \end{bmatrix}^T \end{aligned}$$