Proof that Superposition Implies Rational Homogeneity

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Introduction

In linear system theory, we consider a special class of systems that are deemed *linear* if and only if they obey the properties of superposition and homogeneity:

Homogeneity: A system $H\{\cdot\}$ is homogeneous if for any $H\{x(t)\} = y(t)$, then

$$H\{ax(t)\} = ay(t) \quad a \in \mathbb{R}$$

Superposition: A system obeys superposition if and only if for a class of inputs $x_i(t)$, $i \in \{1, 2, \ldots, N\}$ such that $H\{x_i(t)\} = y_i(t)$, then

$$H\left\{\sum_{i=1}^{N} x_i(t)\right\} = \sum_{i=1}^{N} y_i(t)$$

We introduce a new term *rational homogeneity*. A system $H\{\cdot\}$ is rationally homogeneous if and only if for any $H\{x(t)\} = y(t)$, then

$$H\{ax(t)\} = ay(t) \quad a \in \mathbb{Q}$$

Alternatively, we can write this as

$$H\left\{\frac{m}{n}x(t)\right\} = \frac{m}{n}y(t) \quad m, n \in \mathbb{Z}, n \neq 0$$

Here, we endeavor to show that if a system obeys superposition, then it obeys rational homogeneity. Call the set of all systems that obey superposition S and the set of all systems that obey rational homogeneity RH. We endeavor to prove that for any system $H\{\cdot\}$

$$H\{\cdot\} \in S \Rightarrow H\{\cdot\} \in RH$$

Proof

The formal statement of the problem follows. For any system $H\{\cdot\}$

Given:

$$H\{x_i(t)\} = y_i(t) \quad i \in \{1, 2, \ldots, N\}$$

$$H\left\{\sum_{i=1}^{N} x_i(t)\right\} = \sum_{i=1}^{N} y_i(t)$$

Prove: For any input $x(t)$ such that $H\{x(t)\} = y(t)$

$$H\left\{\frac{m}{n}x(t)\right\} = \frac{m}{n}y(t) \quad m, n \in \mathbb{Z}, n \neq 0$$

Let

$$y_{1/n}(t) = H\left\{\frac{1}{n}x(t)\right\}$$

for some $n \in \mathbb{Z}$, $n \neq 0$. It follows that

$$H\{x(t)\} = H\left\{\sum_{i=1}^{n}\frac{1}{n}x(t)\right\} = \sum_{i=1}^{n} y_{1/n}(t) = ny_{1/n}(t) = nH\left\{\frac{1}{n}x(t)\right\}$$

Rearranging the above gives

$$H\left\{\frac{1}{n}x(t)\right\} = \frac{1}{n}H\{x(t)\}$$

Now, let

$$x_{1/n}(t) = \frac{1}{n}x(t)$$

It follows that

$$H\{x_{1/n}(t)\} = \frac{1}{n}H\{x(t)\}$$

From the given, we have that

$$H\{mx_{1/n}(t)\} = H\left\{\sum_{i=1}^{m} x_{1/n}(t)\right\} = \sum_{i=1}^{m} H\{x_{1/n}(t)\} = mH\{x_{1/n}(t)\}$$

for some $m \in \mathbb{Z}$. Substituting in the definition for $x_{1/n}(t)$ gives

$$H\left\{m\frac{1}{n}x(t)\right\} = m\frac{1}{n}H\{x(t)\}$$

That is

$$H\left\{\frac{m}{n}x(t)\right\} = \frac{m}{n}H\{x(t)\}$$

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