Expected Value of the Rayleigh Random Variable

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We consider the Rayleigh density function, that is, the probability density function of the Rayleigh random variable, given by

$$f_R(r) = \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}}$$

Note that this is radial, so we consider $f_R(r)$ for $r > 0$. We endeavor to find the expectation of this random variable. This is given by the integral

$$E\{R\} = \int_0^\infty r f_R(r) dr = \int_0^\infty \frac{r^2}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} dr$$

I offer three solutions.

Solution with Integration by Parts

First, we will consider an integration by parts approach. Recall

$$\int u \ dv = uv - \int v \ du$$

Careful choice of u and v gives

$$E\{R\} = \frac{1}{\sigma^2} \int_0^\infty \underbrace{r}_{u} \cdot \underbrace{r e^{-\frac{r^2}{2\sigma^2}} dr}_{dv}$$

That is

$$\begin{aligned} u &= r \\ du &= dr \\ v &= -\sigma^2 e^{-\frac{r^2}{2\sigma^2}} \\ dv &= r e^{-\frac{r^2}{2\sigma^2}} dr \end{aligned}$$

This gives

$$E\{R\} = \frac{1}{\sigma^2} \left(-\sigma^2 r e^{-\frac{r^2}{2\sigma^2}} \Big|_0^\infty + \int_0^\infty \sigma^2 e^{-\frac{r^2}{2\sigma^2}} dr \right) = \underbrace{-r e^{-\frac{r^2}{2\sigma^2}} \Big|_0^\infty}_{\alpha} + \underbrace{\int_0^\infty e^{-\frac{r^2}{2\sigma^2}} dr}_{\beta}$$

The expectation is the sum of the two terms α and β. Let us consider α:

$$\alpha = \lim_{r\to\infty} \left(-r e^{-\frac{r^2}{2\sigma^2}} \right) - \lim_{r\to 0} \left(-r e^{-\frac{r^2}{2\sigma^2}} \right)$$

The first limit is indeterminant, but we can see that the exponential term decays to zero faster than the polynomial term grows. Formally, we use L'Hôpital's rule to evaluate the limit:

$$\lim_{r\to\infty} \left(-r e^{-\frac{r^2}{2\sigma^2}} \right) = -\lim_{r\to\infty} \frac{r}{e^{\frac{r^2}{2\sigma^2}}} = -\lim_{r\to\infty} \frac{(d/dr) r}{(d/dr) e^{\frac{r^2}{2\sigma^2}}} = -\lim_{r\to\infty} \frac{1}{(r/\sigma^2) e^{\frac{r^2}{2\sigma^2}}} = 0$$

The second term of the limit can be evaluated by simple substitution:

$$\lim_{r\to 0}\left(-re^{-\frac{r^2}{2\sigma^2}}\right) = -re^{-\frac{r^2}{2\sigma^2}}\Big|_{r=0} = 0$$

Thus,

$$\alpha = 0 - 0 = 0$$

Our problem reduces to,

$$E\{R\} = \int_0^\infty e^{-\frac{r^2}{2\sigma^2}}\,dr = \beta$$

This integral is known and can be easily calculated. By symmetry, it is clear that,

$$\beta = \int_0^\infty e^{-\frac{r^2}{2\sigma^2}}\,dr = \frac{1}{2}\int_{-\infty}^\infty e^{-\frac{r^2}{2\sigma^2}}\,dr$$

Consider now

$$\beta^2 = \left(\frac{1}{2}\int_{-\infty}^\infty e^{-\frac{r_1^2}{2\sigma^2}}\,dr_1\right)\left(\frac{1}{2}\int_{-\infty}^\infty e^{-\frac{r_2^2}{2\sigma^2}}\,dr_2\right) = \frac{1}{4}\int_{-\infty}^\infty\int_{-\infty}^\infty e^{-\frac{r_1^2+r_2^2}{2\sigma^2}}\,dr_1\;dr_2$$

We now convert to polar coordinates. Specifically, $\rho^2 = r_1^2 + r_2^2$. By the Jacobian, we have that $dr_1\;dr_2 = \rho\;d\rho\;d\theta$. This gives

$$\begin{aligned}
\beta^2 &= \frac{1}{4}\int_0^{2\pi}\int_0^\infty e^{-\frac{\rho^2}{2\sigma^2}}\,\rho\;d\rho\;d\theta \\
&= \frac{1}{4}\left(\int_0^{2\pi} d\theta\right)\left(\int_0^\infty \rho e^{-\frac{\rho^2}{2\sigma^2}}\,d\rho\right) \\
&= \frac{1}{4}\cdot 2\pi\int_0^\infty \rho e^{-\frac{\rho^2}{2\sigma^2}}\,d\rho \\
&= \frac{\pi}{2}\left[-\sigma^2 e^{-\frac{\rho^2}{2\sigma^2}}\right]_0^\infty \\
&= \frac{\pi}{2}[0-(-\sigma^2)] = \frac{\pi}{2}\sigma^2
\end{aligned}$$

Thus,

$$\beta = \sqrt{\frac{\pi}{2}\sigma^2} = \sigma\sqrt{\frac{\pi}{2}}$$

And, finally,

$$E\{R\} = \sigma\sqrt{\frac{\pi}{2}}$$

Solution with Fourier Transforms

We note from symmetry that

$$E\{R\} = \int_0^\infty \frac{r^2}{\sigma^2}e^{-\frac{r^2}{2\sigma^2}}\,dr = \frac{1}{2}\int_{-\infty}^\infty \frac{r^2}{\sigma^2}e^{-\frac{r^2}{2\sigma^2}}\,dr$$

It is now evident that

$$E\{R\} = \frac{1}{2}\int_{-\infty}^\infty \left(\frac{r^2}{\sigma^2}e^{-\frac{r^2}{2\sigma^2}}\right)e^{-2\pi(0)r}dr = F\left(\frac{r^2}{2\sigma^2}e^{-\frac{r^2}{2\sigma^2}}\right)(s)\Big|_{s=0}$$

We note that

$$F\left(\frac{1}{2\sigma^2}e^{-\frac{r^2}{2\sigma^2}}\right)(s) = \frac{1}{\sigma}\sqrt{\frac{\pi}{2}}e^{-2\pi^2\sigma^2 s^2}$$

By the derivative theorem, we have that

$$F\left(\frac{r^2}{2\sigma^2}e^{-\frac{r^2}{2\sigma^2}}\right) = \left(\frac{j}{2\pi}\right)^2 \frac{d^2}{ds}F\left(\frac{1}{2\sigma^2}e^{-\frac{r^2}{2\sigma^2}}\right)(s)$$

Differentiating twice gives

$$F\left(\frac{r^2}{2\sigma^2}e^{-\frac{r^2}{2\sigma^2}}\right)(s) = \left(\frac{j}{2\pi}\right)^2 \left(\frac{1}{\sigma}\sqrt{\frac{\pi}{2}}\right)(-4\pi^2\sigma^2)(1-4\pi^2\sigma^2 s^2)e^{-2\pi^2\sigma^2 s^2}$$

Evaluating this expression at $s = 0$ gives

$$\begin{aligned} E\{R\} &= F\left(\frac{r^2}{2\sigma^2}e^{-\frac{r^2}{2\sigma^2}}\right)(s)\Bigg|_{s=0} \\ &= \left(\frac{j}{2\pi}\right)^2 \left(\frac{1}{\sigma}\sqrt{\frac{\pi}{2}}\right)(-4\pi^2\sigma^2)(1-4\pi^2\sigma^2 s^2)e^{-2\pi^2\sigma^2 s^2}\Bigg|_{s=0} \\ &= \sigma\sqrt{\frac{\pi}{2}} \end{aligned}$$

Solution with the Gamma Function

From before, we have,

$$E\{R\} = \int_0^\infty \frac{r^2}{\sigma^2}e^{-\frac{r^2}{2\sigma^2}}dr$$

We now make a variable substitution given by

$$\begin{aligned} s &= \frac{r^2}{2\sigma^2} \qquad \Rightarrow \qquad r = \sigma\sqrt{2s} \\ ds &= \frac{r\ dr}{\sigma^2} \end{aligned}$$

The integral becomes

$$E\{R\} = \sigma\sqrt{2}\int_0^\infty \sqrt{s}\ e^{-s}ds$$

We recognize this as the Gamma Function $\Gamma(x)$[1] evaluated at $x = \frac{3}{2}$:

$$E\{R\} = \sigma\sqrt{2}\int_0^\infty s^{\frac{3}{2}-1}e^{-s}ds = \sigma\sqrt{2}\ \Gamma\left(\frac{3}{2}\right)$$

It is well known that

$$\Gamma(x+1) = x\ \Gamma(x)$$

and that

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

[1] The Gamma Function is defined as

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$$

Thus

$$\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\,\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}$$

and, finally

$$E\{R\} = \sigma\sqrt{\frac{\pi}{2}}$$