Moments of the Standard Normal Probability Density Function

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We seek a closed-form expression for the mth moment of the zero-mean unit-variance normal distribution. That is, given $X \sim \mathcal{N}(0,1)$, we seek a closed-form expression for $\mathrm{E}(X^m)$ in terms of m.

First, we note that all odd moments of the standard normal are zero due to the symmetry of the probability density function. Now, we consider the case where m is even. From the definition of expectation, we have

$$\begin{aligned} \mathrm{E}(X^m) &= \int_{-\infty}^{\infty} x^m \left(\frac{1}{\sqrt{2\pi}} e^{-x^2/2} \right) dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^{m-1} \left(x e^{-x^2/2} \right) dx \end{aligned}$$

We now use integration by parts, taking

$$\begin{aligned} u &= x^{m-1} \\ dv &= x e^{-x^2/2} dx \end{aligned}$$

which gives

$$\begin{aligned} du &= (m-1) x^{m-2} \\ v &= -e^{-x^2/2} \end{aligned}$$

The moment becomes

$$\begin{aligned} \mathrm{E}(X^m) &= \frac{1}{\sqrt{2\pi}} \left(-x^{m-1} e^{-x^2/2} \Big|_{-\infty}^{\infty} + (m-1) \int_{-\infty}^{\infty} x^{m-2} e^{-x^2/2} dx \right) \\ &= \frac{m-1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^{m-2} e^{-x^2/2} dx \\ &= (m-1) \mathrm{E}\left(X^{m-2}\right) \end{aligned}$$

Since $\mathrm{E}\left(X^0\right) = 1$, the recursive expression can be written as

$$\begin{aligned} \mathrm{E}(X^m) &= (m-1)(m-3)\cdots(3)(1) \\ &= \frac{m!}{\prod_{i=2,4,\ldots,m} i} \\ &= \frac{m!}{\prod_{i=1}^{m/2} 2i} \\ &= \frac{m!}{2^{m/2}(m/2)!} \end{aligned}$$

In conclusion, for $X \sim \mathcal{N}(0,1)$, we have that the mth moment is

$$\mathrm{E}(X^m) = \begin{cases} 0 & m \text{ odd} \\ 2^{-m/2} \frac{m!}{(m/2)!} & m \text{ even} \end{cases}$$