# Derivation of the Matched Filter as Highest $SNR$ Linear Filter

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The matched filter is the linear filter, $h$, that maximizes the output signal-to-noise ratio. If we consider the matched filter as a convolution system with impulse response $h$, with input $x$, the output, $y$, is,

$$y[n] = \sum_{k=-\infty}^{\infty} h[n-k]x[k] \tag{1}$$

Though we most often express filters as the impulse response of convolution systems as in Equation 1, it is easiest to think of the matched filter in the context of a system taking the conjugate inner product, which we will see shortly.

We can derive the linear filter that maximimizes output signal-to-noise ratio by invoking a geometric argument. The intuition behind the matched filter relies on correlating the received signal (a vector) with a filter (another vector) that is parallel with the signal, maximizing the cross product. This enhances the signal. When we consider the additive stochastic noise, we have the additional challenge of minimizing the output due to noise by choosing a filter that is orthogonal to the noise.

Let us formally define the problem. We seek a complex-valued $N$-point FIR filter, $g$, such that we maximize the output signal-to-noise ratio, where the output is the conjugate inner product of the filter and the $N$-point observed signal $x$. Our observed signal consists of the deterministic signal $s$ and additive stochastic noise $w$:

$$x[n] = s[n] + w[n], \text{ for } n \in \{0, 1, \ldots, N-1\}$$

Let us define the covariance matrix of the noise, reminding ourselves that this matrix has Hermitian symmetry, a property that will become useful in the derivation:

$$R_w = E\left\{ww^H\right\}$$

where the superscript $^H$ represents the conjugate transpose operation. Let us call our output, $y$, the conjugate inner product of our filter and the observed signal such that,

$$y = \sum_{n=0}^{N-1} g^*[n]x[n] = g^H x = g^H s + g^H n = y_s + y_w$$

We now define the signal-to-noise ratio, which is our objective function, to be the ratio of the power of the output due to the desired signal to the power of the output due to the noise:

$$SNR = \frac{|y_s|^2}{E\left\{|y_w|^2\right\}} \tag{2}$$

We rewrite Equation 2:

$$SNR = \frac{\left|g^H s\right|^2}{E\left\{\left|g^H w\right|^2\right\}} \tag{3}$$

We wish to solve,

$$\max_g SNR$$

Expanding the denominator of our objective function in Equation 3, we have,

$$E\left\{\left|g^H w\right|^2\right\} = E\left\{(g^H w)(g^H w)^H\right\} = g^H E\left\{ww^H\right\} g = g^H R_w g$$

Now, Equation 3 becomes,

$$SNR = \frac{\left|g^H s\right|^2}{g^H R_w g}$$

We will rewrite this expression with some matrix manipulation[1]. The reason for this seemingly counterproductive measure will become evident shortly. Exploiting the Hermitian symmetry of the covariance matrix $R_w$, we can write,

$$SNR = \frac{\left|(R_w^{1/2} g)^H (R_w^{-1/2} s)\right|^2}{(R_w^{1/2} g)^H (R_w^{1/2} g)} \tag{4}$$

We would like to find an upper bound on the expression in Equation 4. To do so, we first recognize a form of the Cauchy-Schwarz inequality:

$$\left|u^H v\right|^2 \leq \left(u^H u\right)\left(v^H v\right) \tag{5}$$

which is to say that the square of the conjugate inner product of two vectors can only be as large as the product of the individual conjugate inner products of the vectors. This concept returns to the intuition behind the matched filter: the upper bound is achieved when the two vectors $u$ and $v$ are parallel. We resume our derivation by expressing the upper bound on our $SNR$ in light of the inequality in Equation 5:

$$SNR = \frac{\left|(R_w^{1/2} g)^H (R_w^{-1/2} s)\right|^2}{(R_w^{1/2} g)^H (R_w^{1/2} g)} \leq \frac{\left[(R_w^{1/2} g)^H (R_w^{1/2} g)\right]\left[(R_w^{-1/2} s)^H (R_w^{-1/2} s)\right]}{(R_w^{1/2} g)^H (R_w^{1/2} g)} \tag{6}$$

Our valiant matrix manipulation has now paid off. We see that the expression for our upper bound in Equation 6 can be greatly simplified:

$$SNR = \frac{\left|(R_w^{1/2} g)^H (R_w^{-1/2} s)\right|^2}{(R_w^{1/2} g)^H (R_w^{1/2} g)} \leq s^H R_w^{-1} s$$

We can achieve this upper bound if we choose,

$$R_w^{1/2} g = \alpha R_w^{-1/2} s$$

where $\alpha$ is an arbitrary real number. To verify this, we plug into our expression for the output $SNR$ in Equation 6:

[1] I encountered a brief treatment of this method in Melvin, Willian L. "A STAP Overview." *IEEE A&E Systems Magazine* Vol. 19, No. 1 (January 2004): 19-35. This article suggested splitting the covariance matrix $R_w = R_w^{1/2} R_w^{1/2}$ and using the upper bound argument.

$$SNR = \frac{\left|\left(R_w^{1/2} g\right)^H \left(R_w^{-1/2} s\right)\right|^2}{\left(R_w^{1/2} g\right)^H \left(R_w^{1/2} g\right)} = \frac{\alpha^2 \left|\left(R_w^{-1/2} s\right)^H \left(R_w^{-1/2} s\right)\right|^2}{\alpha^2 \left(R_w^{-1/2} s\right)^H \left(R_w^{-1/2} s\right)} = \frac{\left|s^H R_w^{-1} s\right|^2}{s^H R_w^{-1} s} = s^H R_w^{-1} s$$

Thus, our optimal matched filter is,

$$g = \alpha R_w^{-1} s$$

We often choose to normalize the expected value of the power of the noise to unity. That is, we constrain,

$$E\left\{|y_w|^2\right\} = 1$$

This constraint implies a value of $\alpha$, for which we can solve:

$$E\left\{|y_w|^2\right\} = \alpha^2 s^H R_w^{-1} s = 1$$

yielding,

$$\alpha = \frac{1}{\sqrt{s^H R_w^{-1} s}}$$

giving us our normalized filter,

$$g = \frac{1}{\sqrt{s^H R_w^{-1} s}} R_w^{-1} s$$

If we care to write the impulse response of the filter $h$ for the convolution system in Equation 1, it is simply the complex conjugate time reversal of $g$, which we can express so,

$$h = \frac{1}{\sqrt{s^H R_w^{-1} s}} \left(R_w^{-1} s^*\right)^-$$