The Number of Digits in $n!$

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Introduction

This paper investigates the function

$$\psi(n) \triangleq \#(n!)$$

where

$$\#(k) \triangleq \text{number of digits in } k \text{ when } k \text{ is expressed in decimal notation}$$

and is defined for $k \in \mathbb{Z}_+$. Specifically, in this paper we will derive a finite-sum closed-form expression for $\psi(n)$ and observe some of its characteristics.

Derivation of $\psi(n)$

It can easily be seen that the number of digits in some natural number k is closely related to its logarithm in base 10. Specifically, we can easily see that

$$\#(k) - 1 \leq \log k < \#(k)$$

where the left relation holds with equality only for integer powers of 10. It can be seen, then, that the number of digits in some natural number k can be written as

$$\#(k) = 1 + \lfloor \log k \rfloor$$

where $\lfloor \cdot \rfloor$ is the greatest integer or floor operator. As we like to deal in natural logarithms, we can rewrite the above as

$$\#(k) = 1 + \left\lfloor \frac{\ln k}{\ln 10} \right\rfloor$$

We know that

$$n! = \prod_{i=1}^{n} i$$

This gives

$$\psi(n) = 1 + \left\lfloor \frac{\ln \prod_{i=1}^{n} i}{\ln 10} \right\rfloor$$

Exploiting the property of logarithms $\ln ab = \ln a + \ln b$, we have

$$\psi(n) = 1 + \left\lfloor \frac{1}{\ln 10} \sum_{i=1}^{n} \ln i \right\rfloor = 1 + \left\lfloor \frac{1}{\ln 10} \sum_{i=0}^{n-1} \ln(i+1) \right\rfloor$$

Characteristics of $\psi(n)$

It is clear that $\psi(n)$, defined for all natural numbers, is monotonically nondecreasing and unbounded. Specifically, $\psi(n_1) \leq \psi(n_2)$ for any $n_1 \leq n_2$ and

$$\lim_{n\to\infty} \psi(n) = \infty$$

We may wonder, however, where $\psi(n) > n$, where $\psi(n) < n$, and where $\psi(n) = n$. From an initial glance, it is clear that $f(1) = 1$ and that for the next few integers n, $\psi(n) < n$. Is there a threshold at which $\psi(n) > n$? The answer is yes. To begin this discussion, let us embark on an inductive path. Assume that there is some n such that

$$\psi(n) > n$$

With this, we will attempt to show that

$$\psi(n+1) > n+1$$

though not without stipulation. Consider this:

$$\begin{aligned}\psi(n+1) &= 1 + \left\lfloor \frac{1}{\ln 10} \sum_{i=1}^{n+1} \ln i \right\rfloor = 1 + \left\lfloor \frac{1}{\ln 10} \sum_{i=1}^{n} \ln i + \frac{\ln(n+1)}{\ln 10} \right\rfloor \\ &\geq 1 + \left\lfloor \frac{1}{\ln 10} \sum_{i=1}^{n} \ln i \right\rfloor + \left\lfloor \frac{\ln(n+1)}{\ln 10} \right\rfloor = \psi(n) + \left\lfloor \frac{\ln(n+1)}{\ln 10} \right\rfloor\end{aligned}$$

In summary,

$$\psi(n+1) \geq \psi(n) + \#(n+1) - 1$$

Thus, given

$$\psi(n) > n$$

we write,

$$\psi(n+1) \geq \psi(n) + \#(n+1) - 1 > n + \#(n+1) - 1$$

Also, we have that

$$n + \#(n+1) - 1 \geq n+1$$

provided that $n+1$ has at least two digits. If $n+1$ has at least two digits, then any $k > n+1$ will also have at least two digits. It follows by transitivity that if

$$\psi(n) > n$$

and $n+1$ has at least two digits, then

$$\psi(n+1) > n+1$$

Thus, if we find some n_0 such that $n_0 + 1$ has two digits and $\psi(n_0) > n_0$, induction dictates that

$$\psi(n) > n \qquad n \geq n_0$$

A few brief calculations show that

$$\psi(n) < n \qquad n \in \{2, 3, \ldots, 21\}$$

and

$$\psi(n) = n \qquad n \in \{1, 22, 23, 24\}$$

Also, we note that $\psi(25) = 26 > 25$ and $25 + 1$ has two digits. Thus, it follows from the inductive analysis above that

$$\psi(n) > n \qquad n \geq 25$$

This means that for $n \geq 25$, $n!$ has more digits than the value of n itself.