Craps: Computing the Distribution of the Pass-Line and Free Odds Bets

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Introduction

Craps is a popular dice-based casino game where multiple players simultaneously compete against the house. The players have considerable choice in a variety of bets, most of which have payoffs that are severely stacked against the player. The definitive pass-line wager, however, is one of the casino's slimmest margins. In the following analysis, we consider the statistical properties of craps' pass-line bet and the related free odds bet.[1]

The Pass-Line Bet

The *pass-line* bet is the core wager in the game of craps. Players post a bet subject to a table minimum. One player then tosses two dice. If the sum of the dice is 7 or 11, all pass-line bets are paid one-to-one and the game is concluded. If the sum of the dice is 2, 3, or 12, the house collects the pass-line bets and the game is concluded. In the event of a 4, 5, 6, 8, 9, or 10, the game continues and the sum of dice in the first throw is stored and referred to as the *point*. The dice are then tossed as many times as necessary until their sum is either the point, in which case the pass-line bets are paid one-to-one, or until their sum is 7, in which case the pass-line bets are collected by the house. Following either of these two scenarios, the game is concluded.[2]

In our first analysis, we consider the distribution of the payoff of a simple pass-line bet. All analysis is for an initial bet of 1. Let us also define the probability mass function of the sum of two dice as $f(x)$:

$$f(x) = \begin{cases} 1/36, & x \in \{2, 12\} \\ 2/36, & x \in \{3, 11\} \\ 3/36, & x \in \{4, 10\} \\ 4/36, & x \in \{5, 9\} \\ 5/36, & x \in \{6, 8\} \\ 6/36, & x = 7 \\ 0, & \text{otherwise} \end{cases} .$$

The probability p of winning the pass-line bet is the probability of rolling a 7 or 11 or rolling a point and

[1] I owe my interest in the game of craps to my father, Dr. Majid Rabbani, who introduced me to the game. It was he who first shared with me his analysis of the pass-line and odds bets.

[2] When the players win the pass-line bet, the house is said to "pay the line."

subsequently rolling the point again before a 7:[3]

$$\begin{aligned} p &= \Pr\{7\} + \Pr\{11\} + \sum_{i\in\{4,5,6,8,9,10\}} \Pr\{i\} \Pr\{i \text{ before } 7\} \\ &= f(7) + f(11) + \sum_{i\in\{4,5,6,8,9,10\}} f(i)\left(\frac{f(i)}{f(i)+f(7)}\right) = \frac{244}{495} = 0.4929. \end{aligned}$$

Let us define the random variable Π as the payoff from the pass-line bet. Its expected value is

$$e = \mathrm{E}[\Pi] = p - (1-p) = 2p - 1 = -\frac{7}{495} = -0.0141.$$

That is, the house has a 1.41% advantage on the pass-line bet. The standard deviation is

$$\sigma = \sqrt{\mathrm{E}[\Pi^2] - (\mathrm{E}[\Pi])^2} = \sqrt{1-e^2}.$$

The Odds Bet

The pass-line bet is enhanced by the *free odds* bet, often called the *odds* bet for short. In the event that a point is rolled and the game advances beyond the first throw, the player is allowed to place an additional wager that pays only if the point is rolled before the 7. This additional bet is called free odds, because, unlike the initial pass-line bet and all other bets in craps, its payoff corresponds to the true odds of the event; i.e., the house has no advantage. That is, if a 4 or 10 is rolled, the payoff is 2:1; if a 5 or 9 is rolled, the payoff is 3:2; and if a 6 or 8 is rolled, the payoff is 6:5.[4] Because the casino does not make revenue on this bet in expectation, it limits the size of the odds bet to a multiple of the pass-line bet. If that factor is m, we say that the table offers m-times odds.

Event	Probability	constant m-odds payoff	variable k-odds payoff
7 or 11	2/9	1	1
2, 3, or 12	1/9	-1	-1
4 or 10 then point before 7	1/18	$2m+1$	$6k+1$
4 or 10 then 7 before point	1/9	$-m-1$	$-3k-1$
5 or 9 then point before 7	4/45	$(3/2)m+1$	$6k+1$
5 or 9 then 7 before point	2/15	$-m-1$	$-4k-1$
6 or 8 then point before 7	25/198	$(6/5)m+1$	$6k+1$
6 or 8 then 7 before point	5/33	$-m-1$	$-5k-1$

Figure 1: Probabilities and payoffs for combined pass-line and odds bet

In order to dilute the negative expectation of the pass-line bet, we will always consider placing the maximum odds bet in the analysis that follows. We refer to this strategy as the combined pass-line and odds bet. Figure 1 shows the distribution of the payoffs for this strategy. Let us call Π^c_m the payoff from the combined pass-line and constant m-odds bet. Because the odds component of this variable has zero

[3] The probability of rolling the point i before a 7 is easily computed using the rules of conditional probability. The only relevant roll is the first roll that is either i or 7:

$$\Pr\{i \text{ before } 7\} = \Pr\{i | i \cup 7\} = \frac{\Pr\{i\}}{\Pr\{i \cup 7\}} = \frac{f(i)}{f(i)+f(7)}.$$

[4] If the fair payoff of a bet is a:b, then the probability of winning is $b/(a+b)$. This forces the expected value of the bet to be zero: $ap - b(1-p) = a\frac{b}{a+b} - b\frac{a}{a+b} = 0$.

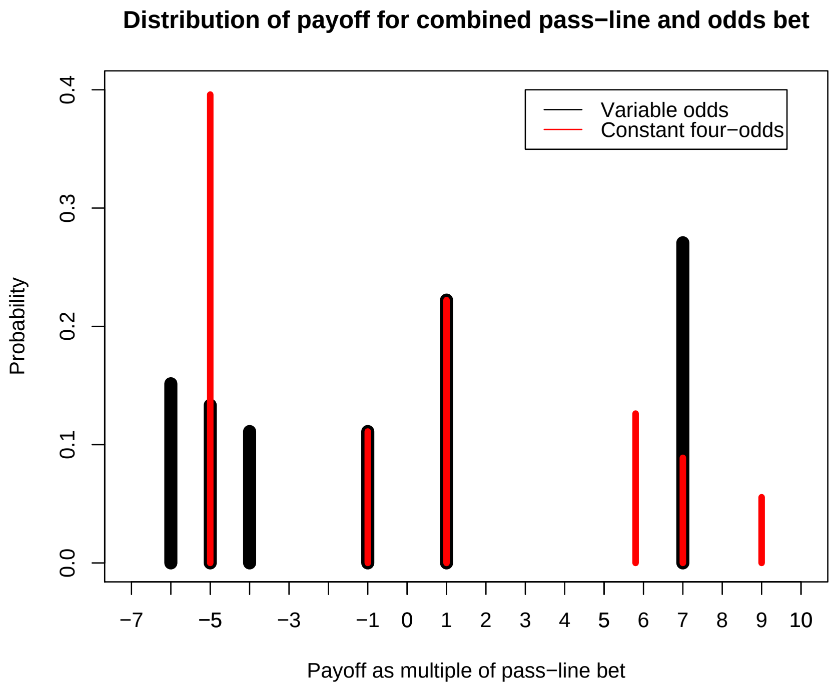

Figure 2: Distribution of payoff for combined pass-line and odds bet

expectation and is additive with the pass-line bet, we must have $\mathrm{E}\left[\Pi_m^c\right] = \mathrm{E}\left[\Pi\right] = e$. The standard deviation of this strategy is computed analytically as $\sigma_m^c = \sqrt{\mathrm{E}\left[\left(\Pi_m^c\right)^2\right] - e^2}$. Strictly in terms of m, this is

$$\sigma_m^c = \sqrt{m^2 + \left(\frac{784}{495}\right) m + (1 - e^2)}.$$

Many casinos vary the maximum odds bet based on the point that is rolled. These variable odds are often found in a 3-4-5 proportion. By allowing a $3k$ wager on 4 and 10, a $4k$ wager on 5 and 9, and a $5k$ wager on 6 and 8, the casino both reduces its risk in the higher variance points and fixes the payoff of a winning odds bet to an easy $6k$, relieving some of the computational burden from the base dealers who pay and collect the bets. The distribution of the payoff Π_k^v in the case of variable k-odds is also shown in Figure 1. As in the case of constant odds, the expected value is still e. The standard deviation is

$$\sigma_k^v = \sqrt{\left(\frac{50}{3}\right) k^2 + \left(\frac{1072}{165}\right) k + (1 - e^2)}.$$

In practice, variable odds are only offered with $k = 1$. As such, we only only consider this case and refer to it simply as "variable odds." The standard deviation for a pass-line and variable odds bet is

$$\sigma^v = \sqrt{\left(\frac{50}{3}\right) (1)^2 + \left(\frac{1072}{165}\right) (1) + (1 - e^2)} = 4.9156.$$

Figure 3 compares the standard deviations of the constant m-odds and variable odds bets. We have quantified two important statistics of our pass-line and odds strategy, which are of great use in the analysis of the relative

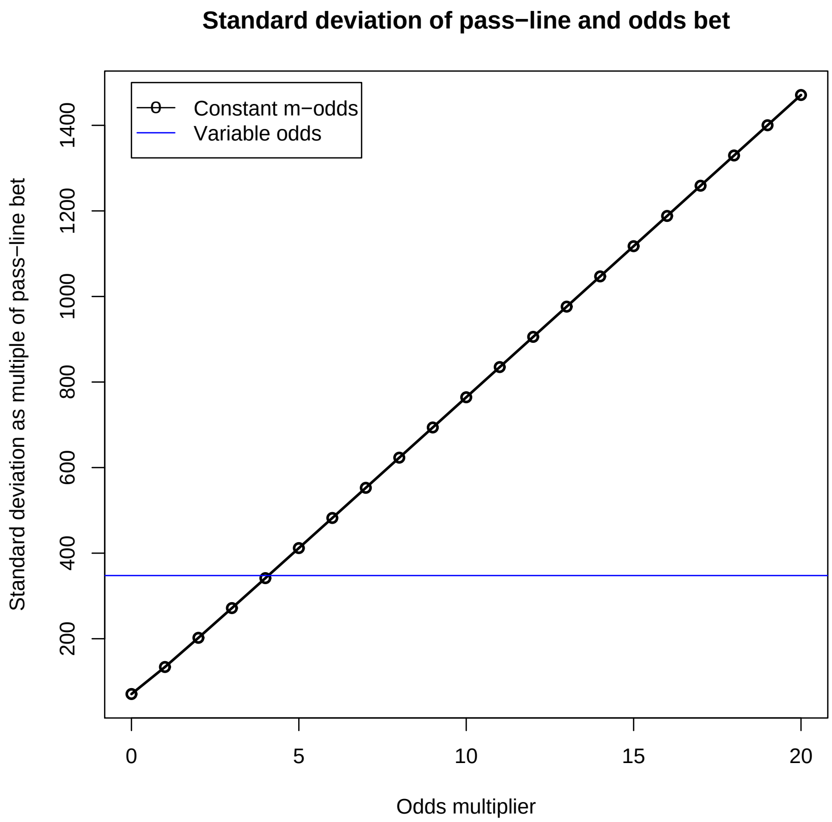

Figure 3: Standard deviation of pass-line and odds bet

merits of a casino game, where we strive for variance in our payoff at the expense of the smallest negative expectation.

Further Analysis

The previous analysis only considers the characteristics of a single pass-line and odds bet. In practice, the player can achieve even greater variance by way of the *come* bet. When a point is achieved and the game advances beyond the first roll or upon any roll thereafter, the player can place a come bet that is independent of the initial wager. The come bet acts as another fresh bet with the same payoff and odds structure as a pass-line bet.

Thus, the variance of the overall game is increased when considering that the player can have an outstanding pass-line bet and multiple come bets, all of which can be forfeited to an unlucky 7. Inversely, the come bet can act as a partial hedge in the first roll after it is placed; in the event of a 7, the player will lose the pass-line bet and its corresponding odds, but will regain some part of the loss with the even-money payout of the come bet.

These further considerations deserve more precise attention. Though the problems may not be mathematically tractable, numerical simulation can provide valuable insight.